

Population Model (Coyotes vs. Rabbits)

(Adapted from a course by the Fractal Foundation)

Exponential Growth

Recurrence relations can be used to model population growth. For example, say we have a population of rabbits and for each new generation, the rabbits produce at a rate of double the current population. This would mean that if we started with 2 rabbits, they would reproduce to have 4 rabbits, giving us a total population of 6 rabbits.

The recurrence relation for this sequence would be given by $p_n = p_{n-1} + 2p_{n-1}$. We would have to provide an initial population, p_1 .

- 1) Using the rule given above, find the number of rabbits in the 8th generation, p_8 , if the initial population $p_1 = 2$.
- 2) Copy your Fibonacci Mathematica program and save it as "Population." Modify your Fibonacci program to see what the population will be in the 20th generation, p_{20} . When you're modifying your program, think about the following things:
 - a) What should **nmax** be?
 - b) Do you need to define both **F[[1]]** and **F[[2]]** before the for loop? Why or why not?
 - c) What needs to be changed in the for loop?
- 3) Plot the population of rabbits using **ListLinePlot** in Mathematica. You can use the command **ListLinePlot[name of sequence]**.

This type of growth is called *Exponential Growth*, and as you can see, the population will get very large, very, very quickly.

- 4) Does this type of population growth make sense? Would you expect to see it in the real world? Why or why not?

The Verhulst Equation

In order to more realistically model population growth, the *Verhulst equation* was introduced in the 19th century. There are several factors that would limit the growth of a population of rabbits in the real world. First, there might not be enough food to support an enormous population. Secondly, there would be predators that would also limit the growth of the population. And rabbits have a limited lifespan, so some of them would die in later generations.

The Verhulst equation takes into account the first two factors by limiting the amount of growth and is given by the following recurrence relation:

$$p_n = p_{n-1} + cp_{n-1} \left(1 - \frac{p_{n-1}}{K}\right)$$

The first term, p_{n-1} , gives the current population. The second term, $cp_{n-1} \left(1 - \frac{p_{n-1}}{K}\right)$, gives the growth or the amount of reproduction given by the current population.

The Verhulst Equation includes something called a *carrying capacity*, given by K , which limits the amount of growth and sets a maximum total population for the rabbits. The growth is also determined by, c , the fertility factor, which would determine how much the rabbits reproduce. Each of these “parameters” would be assigned numeric values for a model.

Example: Let’s say the fertility factor is $c = 0.5$ and the carrying capacity is $K = 200$, and the initial population of rabbits is $p_1 = 100$.

- 1) The model becomes $p_n = p_{n-1} + 0.5p_{n-1} \left(1 - \frac{p_{n-1}}{200}\right)$. Copy the bracket with your exponential growth code from above. Then revise it to find the population in the 10th generation, p_{20} .
- 2) Plot the population of rabbits using a **ListLinePlot** in Mathematica. You can use the command **ListLinePlot[name of sequence]**.
- 3) What is the difference between this plot and the plot from the previous model?

The Logistic Map

The Verhulst Equation can exhibit some very complex behavior depending on the fertility factor, the carrying capacity, and the initial population (c , K , and p_1). Sometimes the behavior has a recognizable pattern, and at other times, it can be completely chaotic.

We will take a look at some of this behavior, but we will look at a simplified version of the equation that is called the “Logistic Map.” The Logistic Map is given below:

$$p_n = cp_{n-1}(1 - p_{n-1})$$

- 1) Describe any differences you notice between the Verhulst Equation and the Logistic Map.
- 2) Describe what the Logistic Map would represent in the real world. What is it modeling? Use the changes in the equation to come to a conclusion.
- 3) Copy and modify your Verhulst Equation code to model the Logistic Map and find all the terms up to p_{65} . (You will need to add an extra statement at the beginning of your program to define c). Start with **$c = 0.2$** ; and $p_1 = 0.5$. Make sure to add the name of the sequence at the end of the code, so you can see the list of terms.
- 4) Plot the population of rabbits using a **ListLinePlot** in Mathematica. You can use the command **ListLinePlot[name of sequence]**.
- 5) Also create a histogram of the values in the sequence by using the command **Histogram[name of sequence, 50]**. (Here the number 50 tells Mathematica how many bars the number range should be split into.)
- 6) Fill in the following table with your observations by changing the **c** value to those given in the table. Use the sequence, the list plot, and the histogram to make observations. Some of the table has already been filled in to give you some hints of what to look for.

Pay attention to what happens to the terms in the sequence. Do they approach a certain number or numbers? You can use the histogram to see if there is more than one number that appears in the sequence frequently, and you can use the sequence to see what those numbers are.

Are these numbers getting larger or smaller as you increase **c** ?

Is the behavior of the numbers in the sequence periodic (meaning: is there a repeated pattern) or is it chaotic? How does the graph behave?

c	How Many Repeated Numbers? Values?	Repeated Numbers Increasing or Decreasing?	Graph Behavior?
0.2			
0.4			
0.6			
0.8			
1			
1.2			
1.4			
1.6			
1.8			
2			
2.2			
2.4			

2.6			Oscillates, dies to one value
2.8			
3			
3.2	2 repeated numbers: 0.799 and 0.513		
3.4		The greater number increased, and the lesser number decreased.	
3.6			
3.8			

- 7) Summarize what you have observed for different values of c .
- 8) Predict the behavior you would expect to see for $c = 3.6625$ and $c = 3.8375$.
- 9) Try inputting the two values of c . What do you observe?

As we can see, the Logistic Map produces some very complex, beautiful, and at times, unexpected, behavior. In fact, if we were to make a scatterplot using the values of c as our x-values and the values from the second column in our table as the y-values, it would produce a figure known as a *bifurcation map*.

- 10) Using the data in your table, make a rough scatterplot by hand.
- 11) Imagine we had the data for all values of c and could connect the points in the scatterplot. Predict what this would look like. You might want to discuss this with your group members before coming to a conclusion.
- 12) Look up “Logistic Map” on Wikipedia and find the actual *bifurcation map*. Is it what you expected? Why or why not?

What do you notice? Does the map agree with the observations you made in your table? Why or why not?