<u>Introduction to Recurrence Relations</u>

<u>The Fibonacci Sequence</u>

A **recurrence relation** is a definition for a sequence of numbers in which the value of each term in the sequence depends on the values of previous terms in the sequence.

One of the most famous examples is the Fibonacci Sequence, which has the following definition: $a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$, $a_2 = 1$. Here, the subscript n is part of the variable a and just tells us the order of the term in the sequence: a_1 is a variable that represents "the first term in the sequence."

What does this mean?
The first two terms in the sequence are a_1 and a_2: 1 and 1. The next term, a_3 can be found by letting $n = 3$ in the definition:

$$a_3 = a_2 + a_1 = 1 + 1 = 2$$

The fourth term would be

$$a_4 = a_3 + a_2 = 2 + 1 = 3$$

The sequence so far is 1, 1, 2, 3.

1) Using the same format as above, find the next four terms in the sequence:

   $a_5 =$

   $a_6 =$

   $a_7 =$

   $a_8 =$

2) Describe, in words, the rule for finding each term in the sequence.

3) Write a Mathematica program that outputs the first n terms in the sequence.
   a) Open Mathematica and create a new notebook.
   b) Save your notebook as "Fibonacci".

c) Define the value of n (for now we'll use 8) by typing: **nmax = 8;**

d) Next we want to create a table to store all the terms in the sequence. We know there will be **nmax**, or 8, terms in the sequence. To begin, we just want to fill the table with zeros, so enter: **F = Table[0,{n,1,nmax}];**
(It is a good idea to run the statement without the semicolon as well so you can see what this table looks like in Mathematica.)

e) We want to tell Mathematica what the first and second terms are in the sequence and store them in the first and second spaces in our table **F**. To indicate a certain space in a table, you have to use the name of the table, and double brackets around the space number "**[[***space number***]].**" Type:
F[[1]] = 1;
F[[2]] = 1;
(It is probably a good idea to now type **F** and run the statements to see how the table has changed.)

f) Now we can define the rest of the spaces in the table using the definition for the Fibonacci sequence and a "for" loop. Type the following:
For[i = 3, i <= nmax, i++,
F[[i]] = F[[i - 1]] + F[[i - 2]];
]
Mathematica will go through the whole numbers starting from 3 all the way to **nmax** (or 8), and it will insert that number in place of **i**. So for **F[[3]]**, it will calculate **F[[2]] + F[[1]]**, then for **F[[4]]**, it will calculate **F[[3]] + F[[2]]**, etc. The **i++** in the for loop tells Mathematica to increase the **i** values by 1. The **i = 3** tells Mathematica the starting value for **i**, and the **i <= nmax** tells Mathematica the ending value.

Why do you think we start the for loop at **i = 3**? Discuss with your group.

g) Next type **F**, so that we can see the final output. Does your output match your calculated values? If not, double check your calculations and double check your program.

The Golden Ratio

One of the most interesting features of the Fibonacci Sequence is the relationship between successive terms. If you look at a sequence of the ratios of each term and its previous term (a_n/a_{n-1}), the sequence approaches a number called the Golden Ratio. This number is so important that it has been assigned its own Greek letter, ϕ, phi.

Let's add on to the Fibonacci program in Mathematica to determine what this number is.

1) First change **nmax = 50;**. This will allow us to compare more terms for the ratio sequence.
2) Now we want to create a new sequence using a process very similar to the one we used for the Fibonacci sequence. First we will identify how many terms are in the sequence. The first term in our ratio sequence will be the second Fibonacci divided by the first, the second term in our ratio sequence will be the third Fibonacci divided by the second, etc. If we continue in this manner, how many terms will there be?....
3) There will be one less than in the Fibonacci sequence or **nmax - 1**. So we can create our table of zeros again and name it **R**. In a new bracket, type:
   R = Table[0,{n,1,nmax-1}];
4) Since we do not have to use a separate rule to define any beginning terms, we can go straight to the for loop:
   For[i = 1, i <= nmax - 1, i++,
   R[[i]] = F[[i+1] / F[[i]];
   ]
   Discuss this code with your group members. Does it make sense? What do each of the separate parts mean?
5) Now type **R** and run the statements to see the sequence of ratios.
6) You may notice that all the ratios are in fraction form, which is not very helpful. It would be much easier to compare decimal values! Mathematica needs to be given a little hint that this is the format we would prefer. Go back to your Fibonacci lines, and change
   F[[1]] = 1;
   F[[2]] = 1;
   to
   F[[1]] = 1.0;
   F[[2]] = 1.0;
   Run both brackets again. This should change both sequences to decimal values. What number is the sequence of ratios approaching?
7) It would also be nice to see a plot of these values. We can do this very easily by typing: **ListLinePlot[R]**. This will plot the values of the sequence as y-values with 1,2,3, etc. as x-values. What is the behavior of the plot?
8) You may notice that some of the points seem to be outside the plot window. You can fix this by adding an option: **ListLinePlot[R, PlotRange->All]**. Try running this again and observe the difference.

The Golden Ratio is said to appear frequently in nature, art, and architecture. Shapes with side lengths whose proportions are equal to the Golden Ratio are supposed to be more aesthetically pleasing.

9) Read some more about the Golden Ratio on Wikipedia, and find 5 facts about it that you find interesting!